Probabilistic model checking with PRISM: overview and recent developments

Marta Kwiatkowska

Department of Computer Science, University of Oxford

ATVA 2013, Hanoi, October 2013

What is probabilistic model checking?

- Probabilistic model checking…
  - is a formal verification technique
    for modelling and analysing systems
    that exhibit probabilistic behaviour
- Formal verification…
  - is the application of rigorous,
    mathematics-based techniques
    to establish the correctness
    of computerised systems

Why formal verification?

- Errors in computerised systems can be costly…

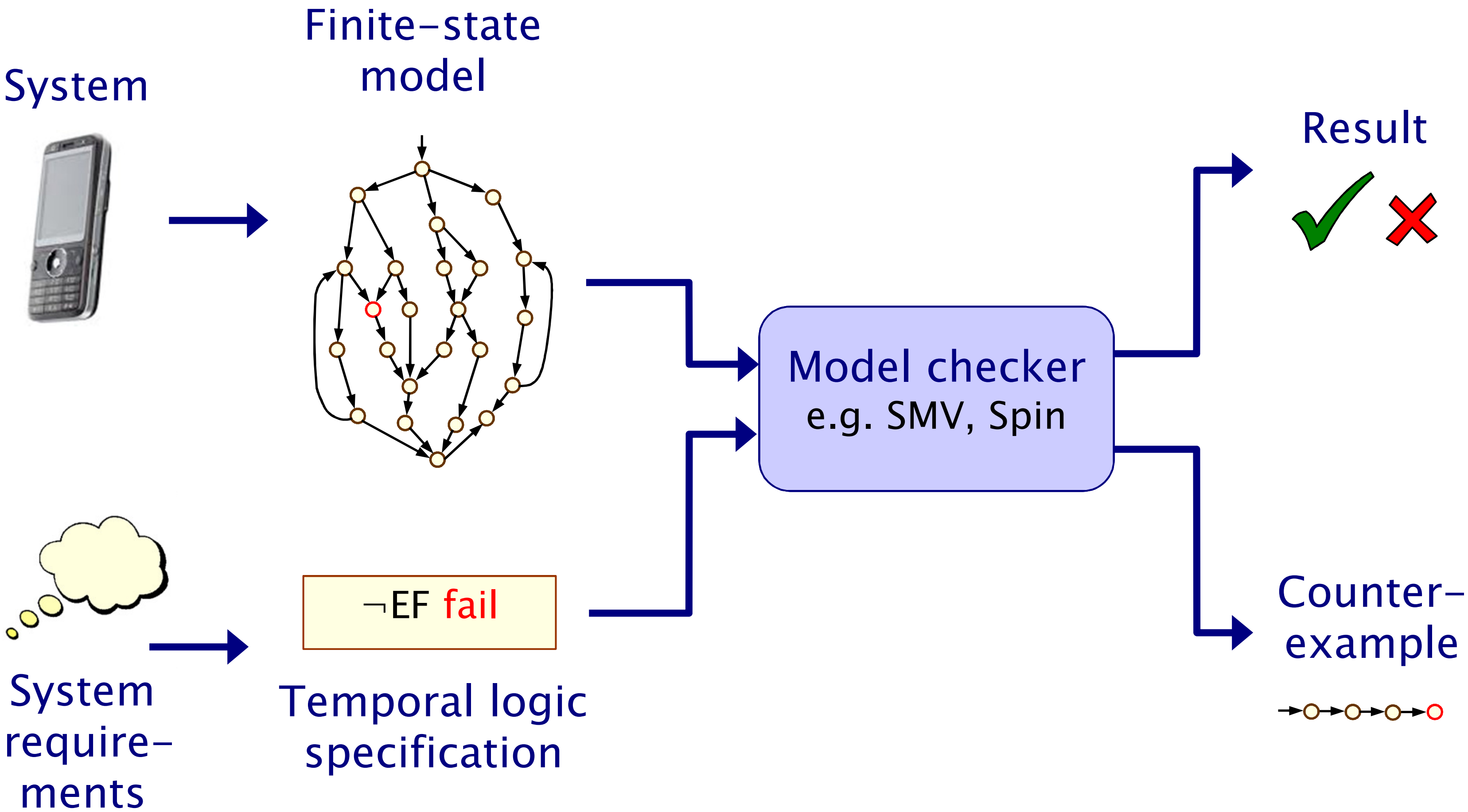
Pentium chip (1994)
Bug found in FPU.
Intel (eventually) offers to replace faulty chips.
Estimated loss: $475m

Ariane 5 (1996)
Self-destructs 37secs into maiden launch.
Cause: uncaught overflow exception.

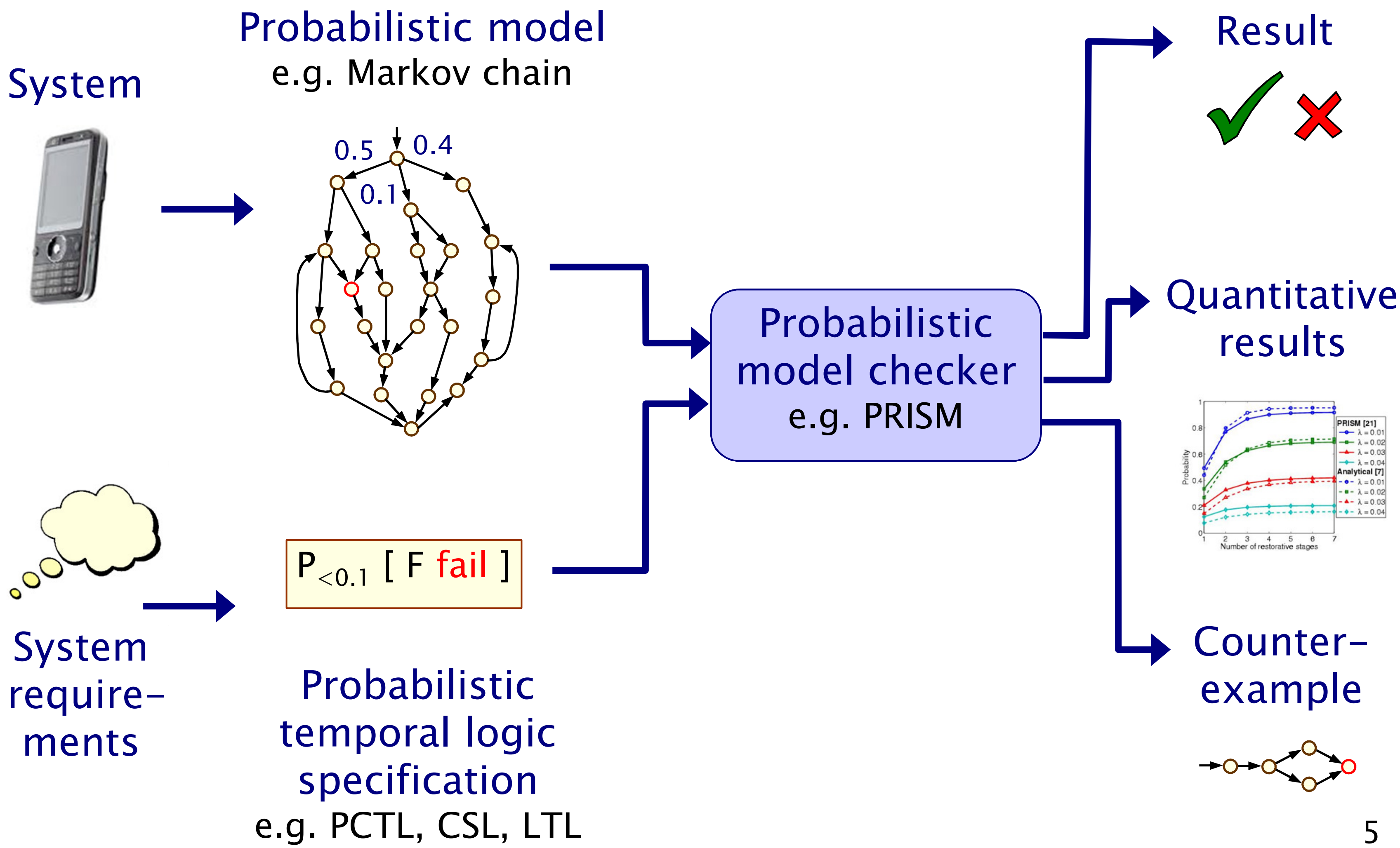
Toyota Prius (2010)
Software "glitch" found in anti-lock braking system.
185,000 cars recalled.

- Why verify?
  - "Testing can only show the presence of errors, not their absence." [Edsger Dijstra]

Model checking



Probabilistic model checking



Why probability?

- Some systems are inherently probabilistic…

- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding

- Examples: real-world protocols featuring randomisation:
  - Randomised back-off schemes
    - CSMA protocol, 802.11 Wireless LAN
  - Random choice of waiting time
    - IEEE1394 Firewire (root contention), Bluetooth (device discovery)
  - Random choice over a set of possible addresses
    - IPv4 Zeroconf dynamic configuration (link-local addressing)
  - Randomised algorithms for anonymity, contract signing, …

Why probability?

- Some systems are inherently probabilistic…
- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding
- To model uncertainty and performance
  - to quantify rate of failures, express Quality of Service
- Examples:
  - computer networks, embedded systems
  - power management policies
  - nano-scale circuitry: reliability through defect-tolerance

Why probability?

- Some systems are inherently probabilistic…
- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding
- To model uncertainty and performance
  - to quantify rate of failures, express Quality of Service
- To model biological processes
  - reactions occurring between large numbers of molecules are naturally modelled in a stochastic fashion

Verifying probabilistic systems

- We are not just interested in correctness
- We want to be able to quantify:
  - security, privacy, trust, anonymity, fairness
  - safety, reliability, performance, dependability
  - resource usage, e.g. battery life
  - and much more…
- Quantitative, as well as qualitative requirements:
  - how reliable is my car's Bluetooth network?
  - how efficient is my phone's power management policy?
  - is my bank's web-service secure?
  - what is the expected long-run percentage of protein X?

Probabilistic models

	Fully probabilistic	Nondeterministic
Discrete time	Discrete-time Markov chains (DTMCs)	Markov decision processes (MDPs)
		Simple stochastic games (SMGs)
Continuous time	Continuous-time Markov chains (CTMCs)	Probabilistic timed automata (PTAs)
		Interactive Markov chains (IMCs)

Probabilistic models

	Fully probabilistic	Nondeterministic
Discrete time	Discrete-time Markov chains (DTMCs)	Markov decision processes (MDPs)
		Simple stochastic games (SMGs)
Continuous time	Continuous-time Markov chains (CTMCs)	Probabilistic timed automata (PTAs)
		Interactive Markov chains (IMCs)

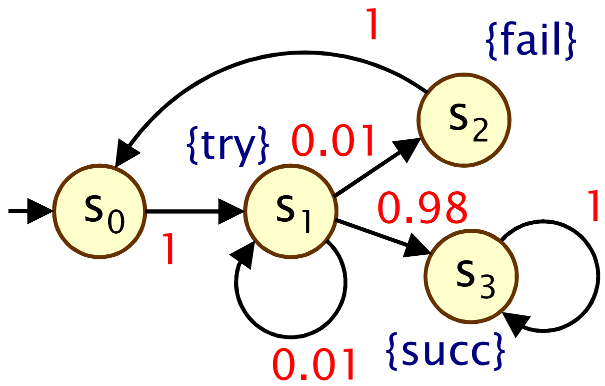
Overview

- Introduction
- Model checking for discrete-time Markov chains (DTMCs)
  - DTMCs: definition, paths & probability spaces
  - PCTL model checking
  - Costs and rewards
  - Case studies: Bluetooth, (CTMC) DNA computing
- PRISM: overview
  - modelling language, properties, GUI, etc
- PRISM: recent developments
  - Multi-objective model checking
  - Parametric models
  - Probabilistic times automata, case study: FireWire
  - Stochastic games, example: smartgrid protocol
- Summary

Discrete-time Markov chains

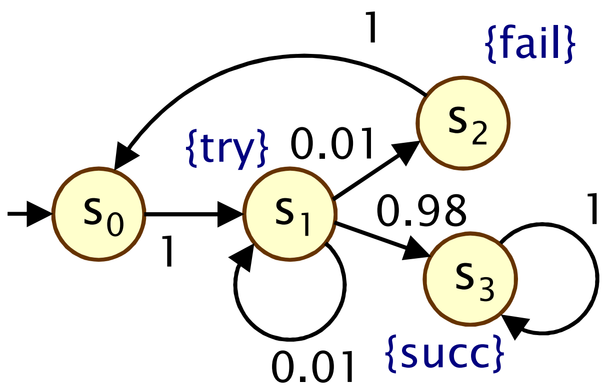
- Discrete-time Markov chains (DTMCs)
  - state-transition systems augmented with probabilities

- States
  - discrete set of states representing possible configurations of the system being modelled
- Transitions
  - transitions between states occur in discrete time-steps
- Probabilities
  - probability of making transitions between states is given by discrete probability distributions



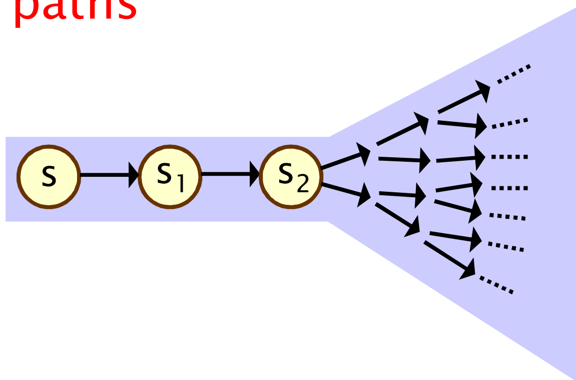
Discrete-time Markov chains

- Formally, a DTMC D is a tuple (S, s_{init}, P, L) where:
  - S is a finite set of states ("state space")
  - $s_{init} \in S$ is the initial state
  - $P : S \times S \rightarrow [0,1]$ is the transition probability matrix where $\Sigma_{s' \in S} P(s,s') = 1$ for all $s \in S$
  - $L : S \rightarrow 2^{AP}$ is function labelling states with atomic propositions
- Note: no deadlock states
  - i.e. every state has at least one outgoing transition
  - can add self loops to represent final/terminating states



Paths and probabilities

- A (finite or infinite) path through a DTMC
  - is a sequence of states $s_0 s_1 s_2 s_3 \ldots$ such that $\mathbf{P}(s_i,s_{i+1}) > 0 \ \forall i$
  - represents an execution (i.e. one possible behaviour) of the system which the DTMC is modelling
- To reason (quantitatively) about this system
  - need to define a probability space over paths
- Intuitively:
  - sample space: Path(s) = set of all infinite paths from a state s
  - events: sets of infinite paths from s
  - basic events: cylinder sets (or "cones")
  - cylinder set C(ω), for a finite path ω = set of infinite paths with the common finite prefix ω
  - for example: $C(ss_1s_2)$



Probability spaces

- Let Ω be an arbitrary non-empty set
- A σ-algebra (or σ-field) on Ω is a family Σ of subsets of Ω closed under complementation and countable union, i.e.:
  - if $A \in \Sigma$, the complement $\Omega \setminus A$ is in Σ
  - if $A_i \in \Sigma$ for $i \in \mathbb{N}$, the union $\cup_i A_i$ is in Σ
  - the empty set $\emptyset$ is in Σ
- Theorem: For any family F of subsets of Ω, there exists a unique smallest σ-algebra on Ω containing F
- Probability space (Ω, Σ, Pr)
  - Ω is the sample space
  - Σ is the set of events: σ-algebra on Ω
  - $Pr : \Sigma \rightarrow [0,1]$ is the probability measure:

    $Pr(\Omega) = 1$ and $Pr(\cup_i A_i) = \Sigma_i Pr(A_i)$ for countable disjoint A_i

Probability space over paths

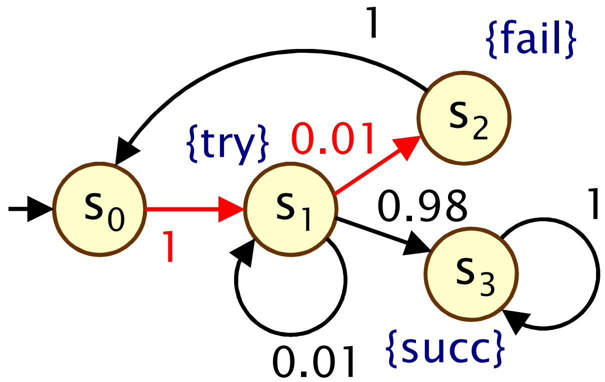
- Sample space Ω = Path(s)
  set of infinite paths with initial state s
- Event set $\Sigma_{Path(s)}$
  - the cylinder set $C(\omega) = \{ \omega' \in Path(s) \mid \omega$ is prefix of $\omega' \}$
  - $\Sigma_{Path(s)}$ is the least σ-algebra on Path(s) containing C(ω) for all finite paths ω starting in s
- Probability measure Pr_s
  - define probability $P_s(\omega)$ for finite path $\omega = ss_1 \ldots s_n$ as:
    - $P_s(\omega) = 1$ if ω has length one (i.e. $\omega = s$)
    - $P_s(\omega) = P(s,s_1) \cdot \ldots \cdot P(s_{n-1},s_n)$ otherwise
    - define $Pr_s(C(\omega)) = P_s(\omega)$ for all finite paths ω
  - Pr_s extends uniquely to a probability measure $Pr_s : \Sigma_{Path(s)} \rightarrow [0,1]$

- See [KSK76] for further details

Probability space – Example

- Paths where sending fails the first time
  - $\omega = s_0 s_1 s_2$
  - $C(\omega)$ = all paths starting $s_0 s_1 s_2 \ldots$
  - $P_{s0}(\omega) = P(s_0,s_1) \cdot P(s_1,s_2)$

    $= 1 \cdot 0.01 = 0.01$
  - $Pr_{s0}(C(\omega)) = P_{s0}(\omega) = 0.01$



- Paths which are eventually successful and with no failures
  - $C(s_0 s_1 s_3) \cup C(s_0 s_1 s_1 s_3) \cup C(s_0 s_1 s_1 s_1 s_3) \cup \ldots$
  - $Pr_{s0}(\, C(s_0 s_1 s_3) \cup C(s_0 s_1 s_1 s_3) \cup C(s_0 s_1 s_1 s_1 s_3) \cup \ldots \,)$

    $= P_{s0}(s_0 s_1 s_3) + P_{s0}(s_0 s_1 s_1 s_3) + P_{s0}(s_0 s_1 s_1 s_1 s_3) + \ldots$

    $= 1 \cdot 0.98 + 1 \cdot 0.01 \cdot 0.98 + 1 \cdot 0.01 \cdot 0.01 \cdot 0.98 + \ldots$

    $= 0.9898989898\ldots$

    $= 98/99$

PCTL

- Temporal logic for describing properties of DTMCs
  - PCTL = Probabilistic Computation Tree Logic [HJ94]
  - essentially the same as the logic pCTL of [ASB+95]
- Extension of (non-probabilistic) temporal logic CTL
  - key addition is probabilistic operator P
  - quantitative extension of CTL's A and E operators
- Example
  - send → $P_{\geq 0.95}$ [true $U^{\leq 10}$ deliver]
  - "if a message is sent, then the probability of it being delivered within 10 steps is at least 0.95"

PCTL syntax

- PCTL syntax:
  - ϕ ::= true | a | $\phi \wedge \phi$ | $\neg\phi$ | $P_{\sim p}$ [ψ] (state formulas)
  - ψ ::= X ϕ | ϕ $U^{\leq k}$ ϕ | ϕ U ϕ (path formulas)

ψ is true with probability ~p

"next"   "bounded until"   "until"

  - where a is an atomic proposition, used to identify states of interest, $p \in [0,1]$ is a probability, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

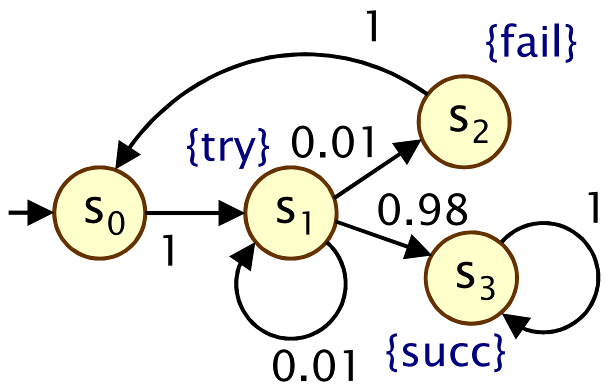
- A PCTL formula is always a state formula
  - path formulas only occur inside the P operator

PCTL semantics for DTMCs

- PCTL formulas interpreted over states of a DTMC
  - $s \vDash \phi$ denotes ϕ is "true in state s" or "satisfied in state s"

- Semantics of (non-probabilistic) state formulas:
  - for a state s of the DTMC (S,s_{init},P,L):
  - $s \vDash a \quad \Leftrightarrow \quad a \in L(s)$
  - $s \vDash \phi_1 \wedge \phi_2 \quad \Leftrightarrow \quad s \vDash \phi_1$ and $s \vDash \phi_2$
  - $s \vDash \neg\phi \quad \Leftrightarrow \quad s \vDash \phi$ is false

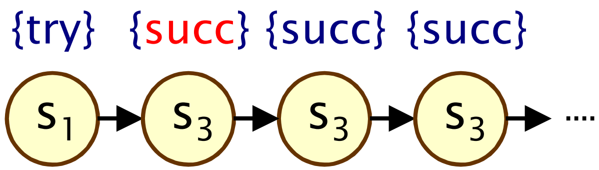
- Examples
  - $s_3 \vDash succ$
  - $s_1 \vDash try \wedge \neg fail$



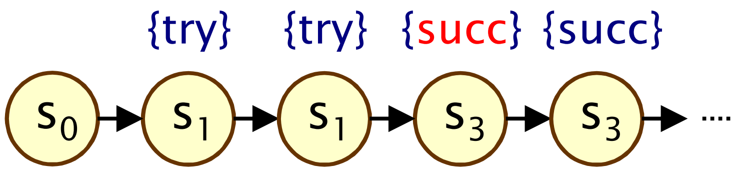
PCTL semantics for DTMCs

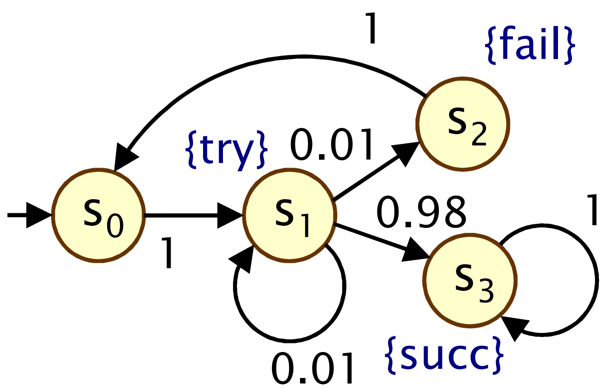
- Semantics of path formulas:
  - for a path $\omega = s_0 s_1 s_2 \ldots$ in the DTMC:
  - $\omega \vDash X\ \phi \quad \Leftrightarrow \quad s_1 \vDash \phi$
  - $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2 \quad \Leftrightarrow \quad \exists i \leq k$ such that $s_i \vDash \phi_2$ and $\forall j < i,\ s_j \vDash \phi_1$
  - $\omega \vDash \phi_1\ U\ \phi_2 \quad \Leftrightarrow \quad \exists k \geq 0$ such that $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2$

- Some examples of satisfying paths:
  - X succ



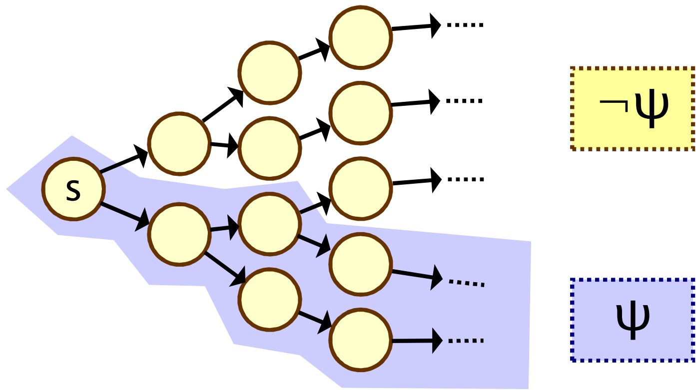
  - ¬fail U succ





PCTL semantics for DTMCs

- Semantics of the probabilistic operator P
  - informal definition: $s \vDash P_{\sim p}$ [ψ] means that "the probability, from state s, that ψ is true for an outgoing path satisfies ~p"
  - example: $s \vDash P_{<0.25}$ [X fail] ⇔ "the probability of atomic proposition fail being true in the next state of outgoing paths from s is less than 0.25"
  - formally: $s \vDash P_{\sim p} [\psi] \Leftrightarrow \mathrm{Prob}(s, \psi) \sim p$
  - where: $\mathrm{Prob}(s, \psi) = \mathrm{Pr}_s \{ \omega \in \mathrm{Path}(s) \mid \omega \vDash \psi \}$
  - (sets of paths satisfying ψ are always measurable [Var85])



Prob(s, ψ) ~ p ?

More PCTL…

- Usual temporal logic equivalences:
  - false ≡ ¬true (false)
  - $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$ (disjunction)
  - $\phi_1 \rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$ (implication)

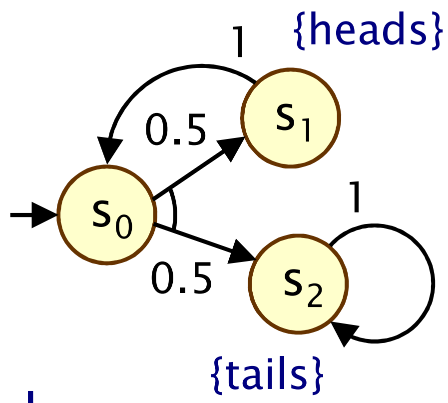
  - $F\ \phi \equiv \Diamond\ \phi \equiv \text{true}\ U\ \phi$ (eventually, “future”)
  - $G\ \phi \equiv \Box\ \phi \equiv \neg(F\ \neg\phi)$ (always, “globally”)
  - bounded variants: $F^{\leq k}\ \phi$, $G^{\leq k}\ \phi$

- Negation and probabilities
  - e.g. $\neg P_{>p}\ [\ \phi_1\ U\ \phi_2\ ] \equiv P_{\leq p}\ [\phi_1\ U\ \phi_2\ ]$
  - e.g. $P_{>p}\ [\ G\ \phi\ ] \equiv P_{<1-p}\ [\ F\ \neg\phi\ ]$

Qualitative vs. quantitative properties

- P operator of PCTL can be seen as a quantitative analogue of the CTL operators A (for all) and E (there exists)

- A PCTL property $P_{\sim p}$ [ψ] is…
  - qualitative when p is either 0 or 1
  - quantitative when p is in the range (0,1)

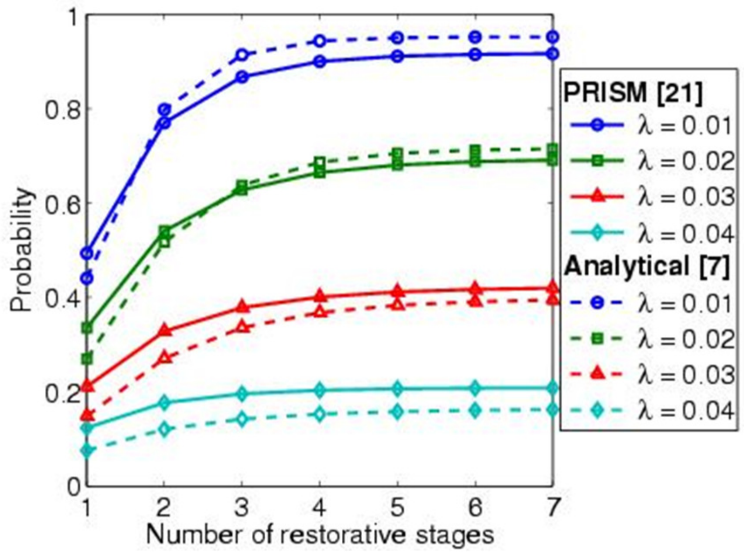


- $P_{>0}$ [F ϕ] is identical to EF ϕ
  - there exists a finite path to a ϕ-state

- $P_{\geq 1}$ [F ϕ] is (similar to but) weaker than AF ϕ
  - e.g. AF "tails" (CTL) ≠ $P_{\geq 1}$ [F "tails"] (PCTL)

Quantitative properties

- Consider a PCTL formula $P_{\sim p}$ [ψ]
  - if the probability is unknown, how to choose the bound p?
- When the outermost operator of a PTCL formula is P
  - we allow the form $P_{=?}$ [ψ]
  - "what is the probability that path formula ψ is true?"
- Model checking is no harder: compute the values anyway
- Useful to spot patterns, trends
- Example
  - $P_{=?}$ [F err/total>0.1]
  - "what is the probability that 10% of the NAND gate outputs are erroneous?"



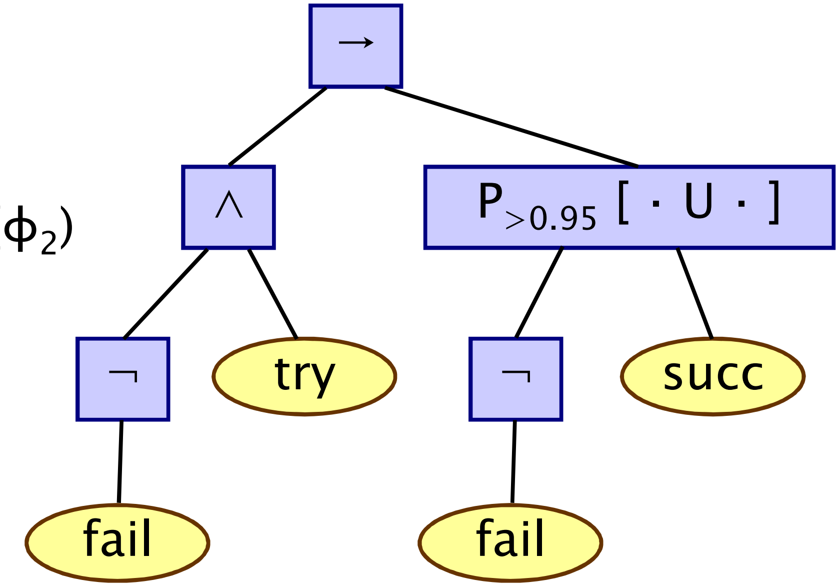
PCTL model checking for DTMCs

- Algorithm for PCTL model checking [CY88,HJ94,CY95]
  - inputs: DTMC $D=(S,s_{init},P,L)$, PCTL formula ϕ
  - output: $Sat(\phi) = \{ s \in S \mid s \models \phi \}$ = set of states satisfying ϕ
- What does it mean for a DTMC D to satisfy a formula ϕ?
  - sometimes, want to check that $s \models \phi \ \forall \ s \in S$, i.e. $Sat(\phi) = S$
  - sometimes, just want to know if $s_{init} \models \phi$, i.e. if $s_{init} \in Sat(\phi)$
- Sometimes, focus on quantitative results
  - e.g. compute result of P=? [F error]
  - e.g. compute result of P=? [$F^{\leq k}$ error] for $0 \leq k \leq 100$

PCTL model checking for DTMCs

- Basic algorithm proceeds by induction on parse tree of ϕ
  - example: $\phi = (\neg\text{fail} \wedge \text{try}) \rightarrow P_{>0.95}\ [\ \neg\text{fail}\ U\ \text{succ}\ ]$

- For the non-probabilistic operators:
  - $\text{Sat}(\text{true}) = S$
  - $\text{Sat}(a) = \{\ s \in S \mid a \in L(s)\ \}$
  - $\text{Sat}(\neg\phi) = S \setminus \text{Sat}(\phi)$
  - $\text{Sat}(\phi_1 \wedge \phi_2) = \text{Sat}(\phi_1) \cap \text{Sat}(\phi_2)$



- For the $P_{\sim p}\ [\ \psi\ ]$ operator
  - need to compute the probabilities $\text{Prob}(s, \psi)$ for all states $s \in S$
  - focus here on "until" case: $\psi = \phi_1\ U\ \phi_2$

PCTL until for DTMCs

- Computation of probabilities Prob(s, ϕ_1 U ϕ_2) for all s ∈ S
- First, identify all states where the probability is 1 or 0
  - S^{yes} = Sat($P_{\geq 1}$ [ϕ_1 U ϕ_2])
  - S^{no} = Sat($P_{\leq 0}$ [ϕ_1 U ϕ_2])
- Then solve linear equation system for remaining states

- We refer to the first phase as "precomputation"
  - two algorithms: Prob0 (for S^{no}) and Prob1 (for S^{yes})
  - algorithms work on underlying graph (probabilities irrelevant)
- Important for several reasons
  - reduces the set of states for which probabilities must be computed numerically (which is more expensive)
  - gives exact results for the states in S^{yes} and S^{no} (no round-off)
  - for $P_{\sim p}[\cdot]$ where p is 0 or 1, no further computation required

PCTL until – Linear equations

- Probabilities Prob(s, ϕ_1 U ϕ_2) can now be obtained as the unique solution of the following set of linear equations:

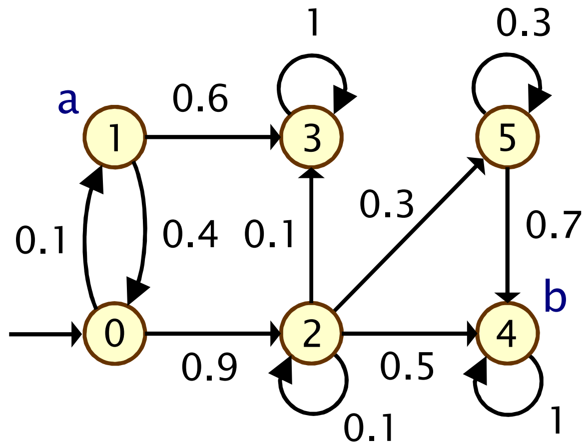
$$\text{Prob}(s, \phi_1 \text{ U } \phi_2) = \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ \sum_{s' \in S} P(s,s') \cdot \text{Prob}(s', \phi_1 \text{ U } \phi_2) & \text{otherwise} \end{cases}$$

  – can be reduced to a system in $|S^?|$ unknowns instead of $|S|$ where $S^? = S \setminus (S^{yes} \cup S^{no})$

- This can be solved with (a variety of) standard techniques
  – direct methods, e.g. Gaussian elimination
  – iterative methods, e.g. Jacobi, Gauss-Seidel, … (preferred in practice due to scalability)

PCTL until – Example

- Example: $P_{>0.8}$ [¬a U b]

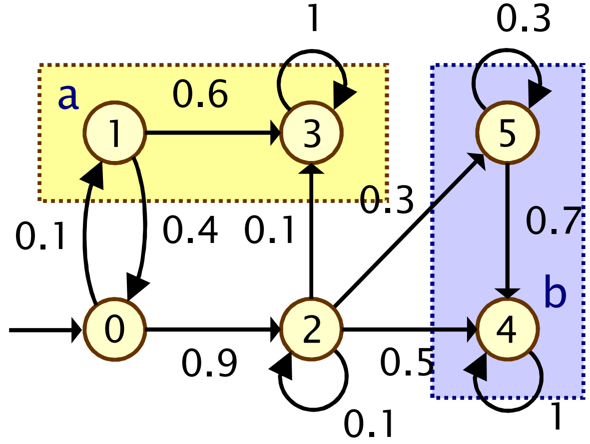


PCTL until – Example

- Example: $P_{>0.8}$ [¬a U b]

S^{no} =

Sat($P_{\leq 0}$ [¬a U b])



S^{yes} =

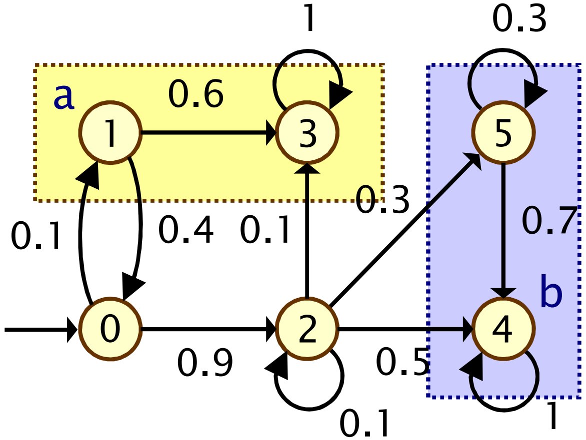
Sat($P_{\geq 1}$ [¬a U b])

PCTL until – Example

- Example: $P_{>0.8}$ [¬a U b]
- Let x_s = Prob(s, ¬a U b)
- Solve:

S^{no} =

Sat($P_{\leq 0}$ [¬a U b])



S^{yes} =

Sat($P_{\geq 1}$ [¬a U b])

$x_4 = x_5 = 1$

$x_1 = x_3 = 0$

$x_0 = 0.1x_1+0.9x_2 = 0.8$

$x_2 = 0.1x_2+0.1x_3+0.3x_5+0.5x_4 = 8/9$

Prob(¬a U b) = x = [0.8, 0, 8/9, 0, 1, 1]

Sat($P_{>0.8}$ [¬a U b]) = { s_2,s_4,s_5 }

PCTL model checking - Summary

- Computation of set Sat(Φ) for DTMC D and PCTL formula Φ
  - recursive descent of parse tree
  - combination of graph algorithms, numerical computation
- Probabilistic operator P:
  - X Φ : one matrix-vector multiplication, $O(|S|^2)$
  - $\Phi_1 \ U^{\leq k} \ \Phi_2$: k matrix-vector multiplications, $O(k|S|^2)$
  - $\Phi_1 \ U \ \Phi_2$: linear equation system, at most |S| variables, $O(|S|^3)$
- Complexity:
  - linear in |Φ| and polynomial in |S|

Limitations of PCTL

- PCTL, although useful in practice, has limited expressivity
  - essentially: probability of reaching states in X, passing only through states in Y (and within k time-steps)

- More expressive logics can be used, for example:
  - LTL [Pnu77] – (non-probabilistic) linear-time temporal logic
  - PCTL* [ASB+95,BdA95] – which subsumes both PCTL and LTL
  - both allow path operators to be combined
  - (in PCTL, $P_{\sim p}$ [...] always contains a single temporal operator)
  - (not covered in this lecture)

- Another direction: extend DTMCs with costs and rewards…

Costs and rewards

- We augment DTMCs with rewards (or, conversely, costs)
  – real-valued quantities assigned to states and/or transitions
  – these can have a wide range of possible interpretations

- Some examples:
  – elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit, …

- Costs? or rewards?
  – mathematically, no distinction between rewards and costs
  – when interpreted, we assume that it is desirable to minimise costs and to maximise rewards
  – we will consistently use the terminology “rewards” regardless

Reward-based properties

- Properties of DTMCs augmented with rewards
  - allow a wide range of quantitative measures of the system
  - basic notion: expected value of rewards
  - formal property specifications will be in an extension of PCTL

- More precisely, we use two distinct classes of property…

- Instantaneous properties
  - the expected value of the reward at some time point

- Cumulative properties
  - the expected cumulated reward over some period

DTMC reward structures

- For a DTMC (S,s_{init},P,L), a reward structure is a pair $(\underline{\rho},\iota)$
  - $\underline{\rho} : S \rightarrow \mathbb{R}_{\geq 0}$ is the state reward function (vector)
  - $\iota : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is the transition reward function (matrix)

- Example (for use with instantaneous properties)
  - "size of message queue": $\underline{\rho}$ maps each state to the number of jobs in the queue in that state, ι is not used

- Examples (for use with cumulative properties)
  - "time-steps": $\underline{\rho}$ returns 1 for all states and ι is zero (equivalently, $\underline{\rho}$ is zero and ι returns 1 for all transitions)
  - "number of messages lost": $\underline{\rho}$ is zero and ι maps transitions corresponding to a message loss to 1
  - "power consumption": $\underline{\rho}$ is defined as the per-time-step energy consumption in each state and ι as the energy cost of each transition

PCTL and rewards

- Extend PCTL to incorporate reward-based properties
  - add an R operator, which is similar to the existing P operator

expected reward is ~r

  - $\phi ::= \ldots \mid P_{\sim p}\,[\, \psi \,] \mid R_{\sim r}\,[\, I^{=k} \,] \mid R_{\sim r}\,[\, C^{\leq k} \,] \mid R_{\sim r}\,[\, F\ \phi \,]$

"instantaneous" "cumulative" "reachability"

  - where $r \in \mathbb{R}_{\geq 0}$, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- $R_{\sim r}\,[\, \cdot \,]$ means "the expected value of $\cdot$ satisfies $\sim r$"

Types of reward formulas

- Instantaneous: $R_{\sim r}$ [$I^{=k}$]
  - "the expected value of the state reward at time-step k is ~r"
  - e.g. "the expected queue size after exactly 90 seconds"
- Cumulative: $R_{\sim r}$ [$C^{\leq k}$]
  - "the expected reward cumulated up to time-step k is ~r"
  - e.g. "the expected power consumption over one hour"
- Reachability: $R_{\sim r}$ [F ϕ]
  - "the expected reward cumulated before reaching a state satisfying ϕ is ~r"
  - e.g. "the expected time for the algorithm to terminate"

Reward formula semantics

- Formal semantics of the three reward operators
  - based on random variables over (infinite) paths

- Recall:
  - $s \vDash P_{\sim p} [\ \psi\ ] \ \Leftrightarrow\ Pr_s \{\ \omega \in Path(s) \mid \omega \vDash \psi\ \} \sim p$

- For a state s in the DTMC:
  - $s \vDash R_{\sim r} [\ I^{=k}\ ] \ \Leftrightarrow\ Exp(s, X_{I=k}) \sim r$
  - $s \vDash R_{\sim r} [\ C^{\leq k}\ ] \ \Leftrightarrow\ Exp(s, X_{C \leq k}) \sim r$
  - $s \vDash R_{\sim r} [\ F\ \Phi\ ] \ \Leftrightarrow\ Exp(s, X_{F\Phi}) \sim r$

where: $Exp(s, X)$ denotes the expectation of the random variable $X : Path(s) \rightarrow \mathbb{R}_{\geq 0}$ with respect to the probability measure Pr_s

Reward formula semantics

- Definition of random variables:
  - for an infinite path $\omega = s_0 s_1 s_2 \ldots$

$$X_{I=k}(\omega) = \underline{\rho}(s_k)$$

$$X_{C\leq k}(\omega) = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{i=0}^{k-1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

$$X_{F\phi}(\omega) = \begin{cases} 0 & \text{if } s_0 \in Sat(\phi) \\ \infty & \text{if } s_i \notin Sat(\phi) \text{ for all } i \geq 0 \\ \sum_{i=0}^{k_\phi - 1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

  - where $k_\phi = \min\{ j \mid s_j \vDash \phi \}$

Model checking reward properties

- Instantaneous: $R_{\sim r}$ [$I^{=k}$]
- Cumulative: $R_{\sim r}$ [$C^{\leq t}$]
  - variant of the method for computing bounded until probabilities
  - solution of recursive equations

- Reachability: $R_{\sim r}$ [F ϕ]
  - similar to computing until probabilities
  - precomputation phase (identify infinite reward states)
  - then reduces to solving a system of linear equation

- For more details, see e.g. [KNP07a]

PCTL model checking summary…

- Introduced probabilistic model checking for DTMCs
  - discrete time and probability only
  - PCTL model checking via linear equation solving
  - LTL also supported, via automata-theoretic methods
- Continuous-time Markov chains (CTMCs)
  - discrete states, continuous time
  - temporal logic CSL
  - model checking via uniformisation, a discretisation of the CTMC
- Markov decision processes (MDPs)
  - add nondeterminism to DTMCs
  - PCTL, LTL and PCTL* supported
  - model checking via linear programming

PRISM

- PRISM: Probabilistic symbolic model checker
  - developed at Birmingham/Oxford University, since 1999
  - free, open source software (GPL), runs on all major OSs
- Construction/analysis of probabilistic models…
  - discrete-time Markov chains, continuous-time Markov chains, Markov decision processes, probabilistic timed automata, stochastic multi-player games, …
- Simple but flexible high-level modelling language
  - based on guarded commands; see later…
- Many import/export options, tool connections
  - in: (Bio)PEPA, stochastic π-calculus, DSD, SBML, Petri nets, …
  - out: Matlab, MRMC, INFAMY, PARAM, …

PRISM…

- Model checking for various temporal logics…
  - PCTL, CSL, LTL, PCTL*, rPATL, CTL, …
  - quantitative extensions, costs/rewards, …
- Various efficient model checking engines and techniques
  - symbolic methods (binary decision diagrams and extensions)
  - explicit-state methods (sparse matrices, etc.)
  - statistical model checking (simulation-based approximations)
  - and more: symmetry reduction, quantitative abstraction refinement, fast adaptive uniformisation, ...
- Graphical user interface
  - editors, simulator, experiments, graph plotting
- See: http://www.prismmodelchecker.org/
  - downloads, tutorials, case studies, papers, …

PRISM modelling language

- Simple, textual, state-based modelling language
  - used for all probabilistic models supported by PRISM
  - based on Reactive Modules [AH99]
- Language basics
  - system built as parallel composition of interacting modules
  - state of each module given by finite-ranging variables
  - behaviour of each module specified by guarded commands
    - annotated with probabilities/rates and (optional) action label
  - transitions are associated with state-dependent probabilities
  - interactions between modules through synchronisation

[send] (s=2) -> p_{loss} : (s'=3)&(lost'=lost+1) + (1-p_{loss}) : (s'=4);

action ⟷ guard ⟷ probability ⟷ update ⟷ probability ⟷ update

Simple example

```
dtmc

module M1
    x : [0..3] init 0;
    [a] x=0 -> (x'=1);
    [b] x=1 -> 0.5 : (x'=2) + 0.5 : (x'=3);
endmodule

module M2
    y : [0..3] init 0;
    [a] y=0 -> (y'=1);
    [b] y=1 -> 0.4 : (y'=2) + 0.6 : (y'=3);
endmodule
```

Costs and rewards

- We augment models with rewards (or, conversely, costs)
  - real-valued quantities assigned to states and/or transitions
  - these can have a wide range of possible interpretations
- Some examples:
  - elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit, …
- Costs? or rewards?
  - mathematically, no distinction between rewards and costs
  - when interpreted, we assume that it is desirable to minimise costs and to maximise rewards
  - we consistently use the terminology "rewards" regardless
- Properties (see later)
  - reason about expected cumulative/instantaneous reward

Rewards in the PRISM language

```
rewards "total_queue_size"
  true : queue1+queue2;
endrewards
```

(instantaneous, state rewards)

```
rewards "time"
  true : 1;
endrewards
```

(cumulative, state rewards)

```
rewards "dropped"
  [receive] q=q_max : 1;
endrewards
```

(cumulative, transition rewards)
(q = queue size, q_max = max. queue size, receive = action label)

```
rewards "power"
  sleep=true : 0.25;
  sleep=false : 1.2 * up;
  [wake] true : 3.2;
endrewards
```

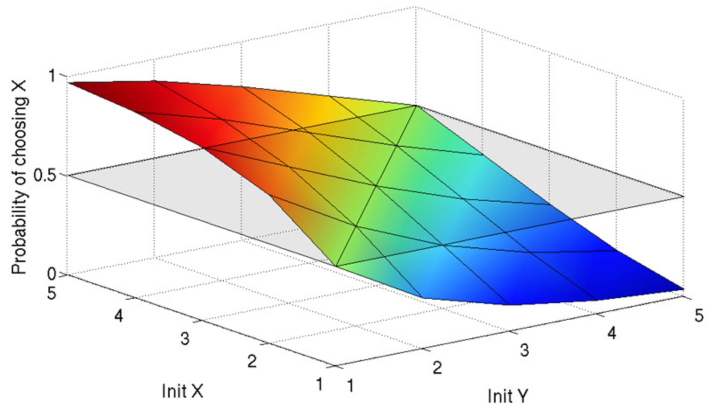
(cumulative, state/trans. rewards)
(up = num. operational components, wake = action label)

PRISM – Property specification

- Temporal logic-based property specification language
  - subsumes PCTL, CSL, probabilistic LTL, PCTL*, …

- Simple examples:
  - $P_{\leq 0.01}$ [F "crash"] – "the probability of a crash is at most 0.01"
  - $S_{>0.999}$ ["up"] – "long-run probability of availability is >0.999"

- Usually focus on quantitative (numerical) properties:
  - $P_{=?}$ [F "crash"]
    "what is the probability of a crash occurring?"
  - then analyse trends in quantitative properties as system parameters vary



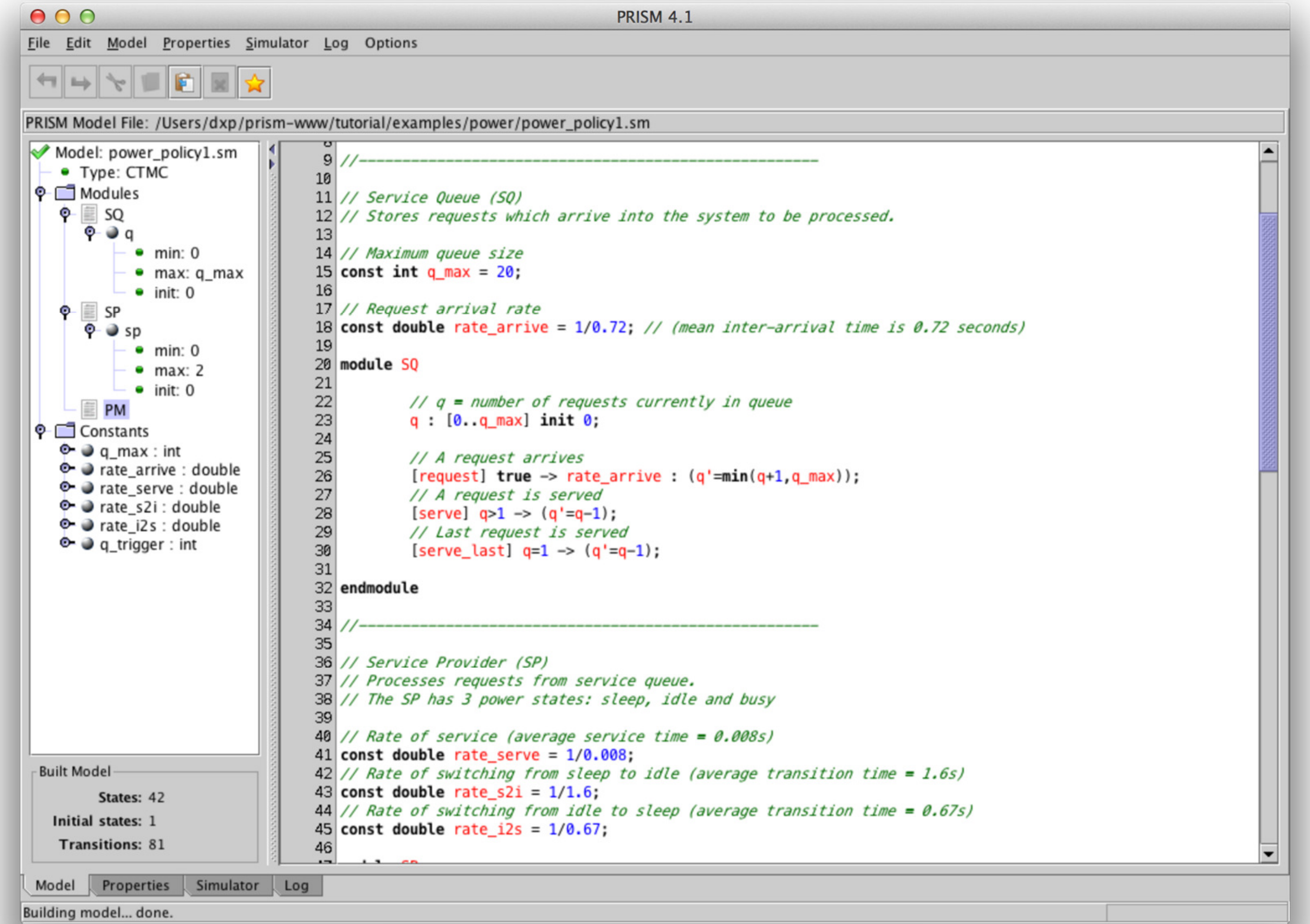
PRISM – Property specification

- Properties can combine numerical + exhaustive aspects
  - $P_{max=?}$ [$F^{\leq 10}$ "fail"] – "worst-case probability of a failure occurring within 10 seconds, for any possible scheduling of system components"
  - $P_{=?}$ [$G^{\leq 0.02}$!"deploy" {"crash"}{max}] – "the maximum probability of an airbag failing to deploy within 0.02s, from any possible crash scenario"

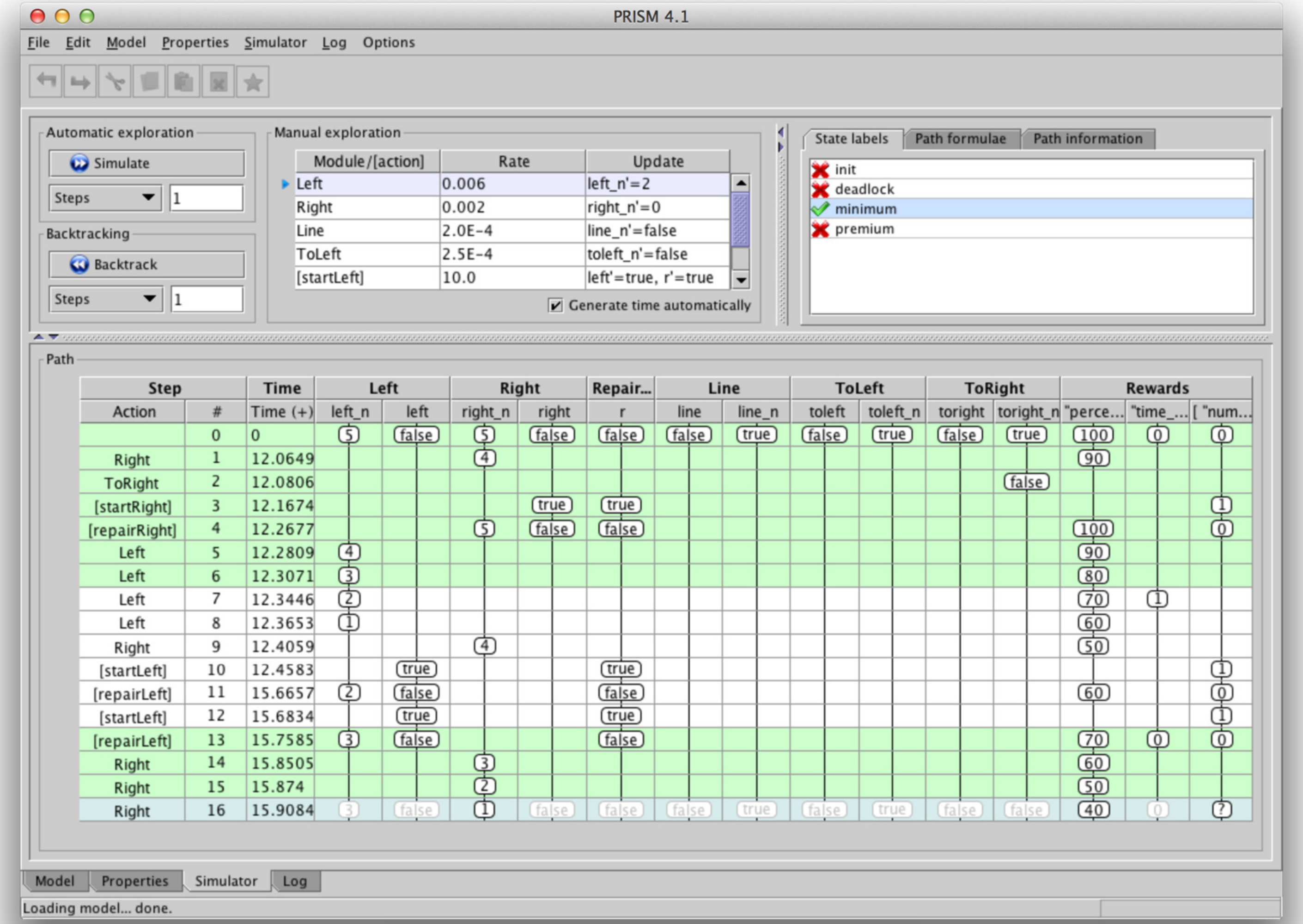
- Reward-based properties (rewards = costs = prices)
  - $R_{\{\text{"time"}\}=?}$ [F "end"] – "expected algorithm execution time"
  - $R_{\{\text{"energy"}\}max=?}$ [$C^{\leq 7200}$] – "worst-case expected energy consumption during the first 2 hours"

- Properties can be combined with e.g. arithmetic operators
  - e.g. $P_{=?}$ [F $fail_1$] / $P_{=?}$ [F $fail_{any}$] – "conditional failure prob."

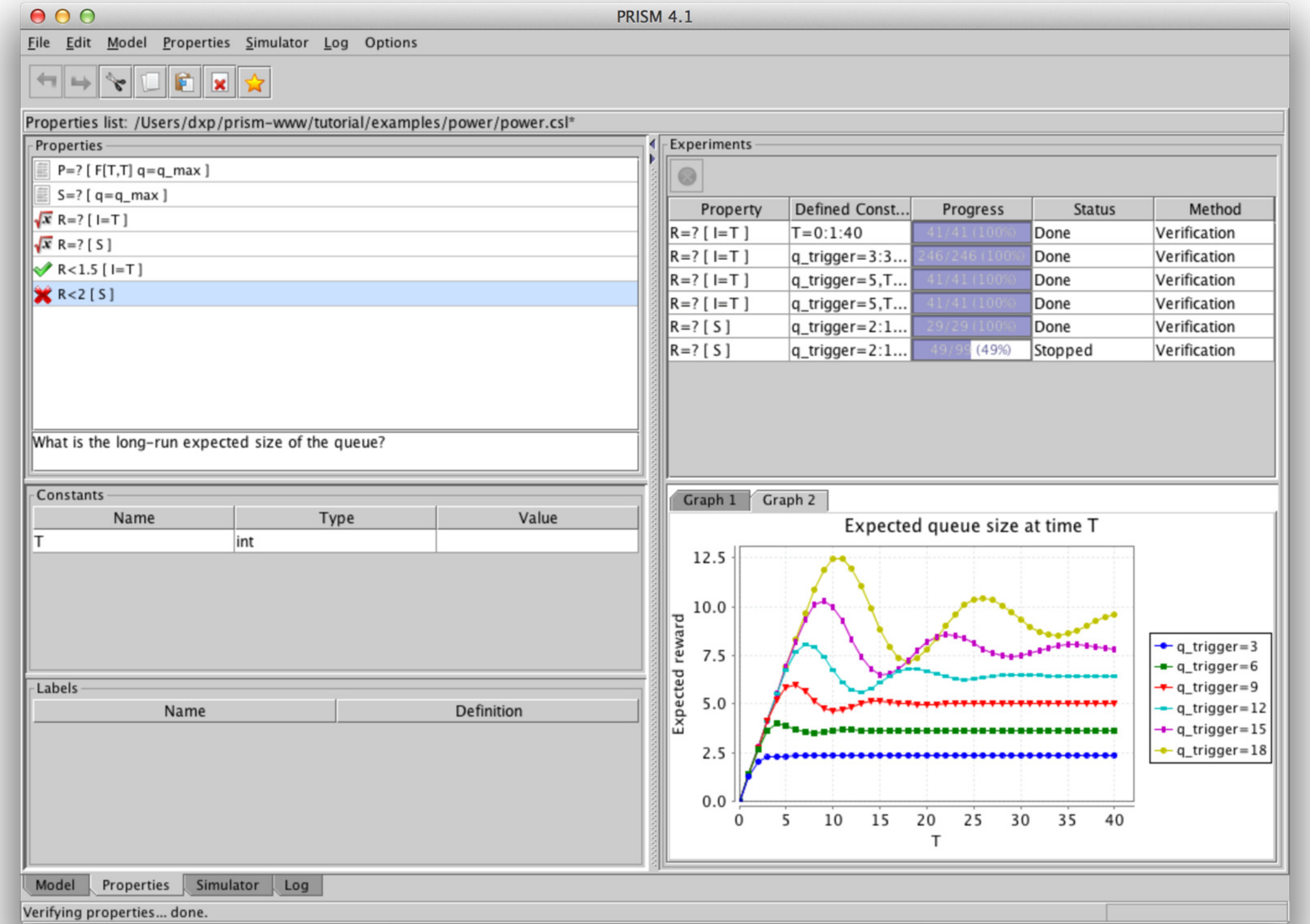
PRISM GUI: Editing a model



PRISM GUI: The Simulator



PRISM GUI: Model checking and graphs



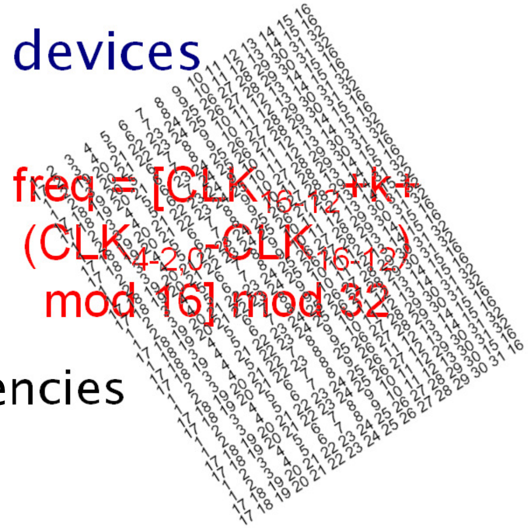
PRISM – Case studies

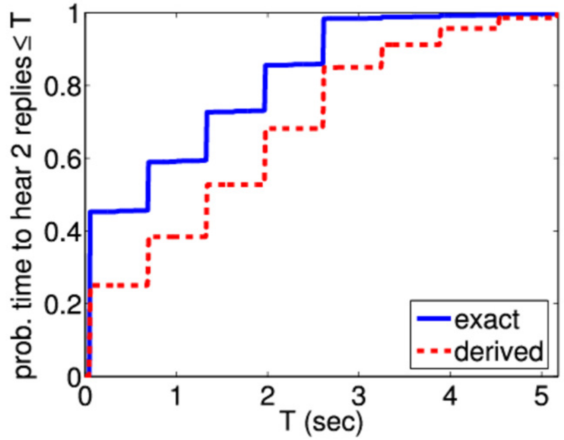
- Randomised distributed algorithms
  - consensus, leader election, self-stabilisation, …
- Randomised communication protocols
  - Bluetooth, FireWire, Zeroconf, 802.11, Zigbee, gossiping, …
- Security protocols/systems
  - contract signing, anonymity, pin cracking, quantum crypto, …
- Biological systems
  - cell signalling pathways, DNA computation, …
- Planning & controller synthesis
  - robotics, dynamic power management, …
- Performance & reliability
  - nanotechnology, cloud computing, manufacturing systems, …

- See: www.prismmodelchecker.org/casestudies

Case study: Bluetooth

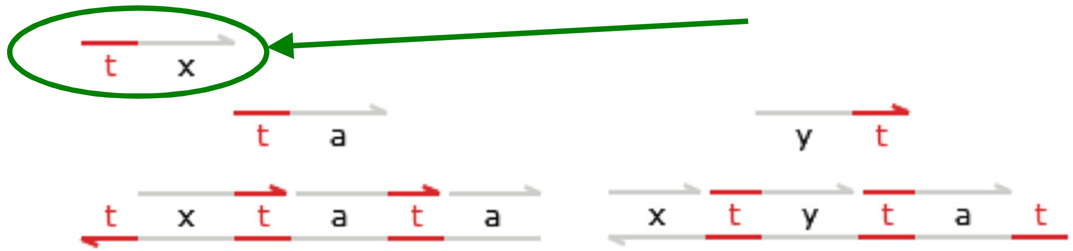
- Device discovery between pair of Bluetooth devices
  - performance essential for this phase
- Complex discovery process
  - two asynchronous 28-bit clocks
  - pseudo-random hopping between 32 frequencies
  - random waiting scheme to avoid collisions
  - 17,179,869,184 initial configurations (too many to sample effectively)
- Probabilistic model checking
  - e.g. "worst-case expected discovery time is at most 5.17s"
  - e.g. "probability discovery time exceeds 6s is always < 0.001"
  - shows weaknesses in simplistic analysis





Case study: DNA programming

- DNA: easily accessible, cheap to synthesise information processing material
- DNA Strand Displacement language, induces CTMC models
  - for designing DNA circuits [Cardelli, Phillips, et al.]
  - accompanying software tool for analysis/simulation
  - now extended to include auto-generation of PRISM models
- Transducer: converts input <t^ x> into output <y t^>

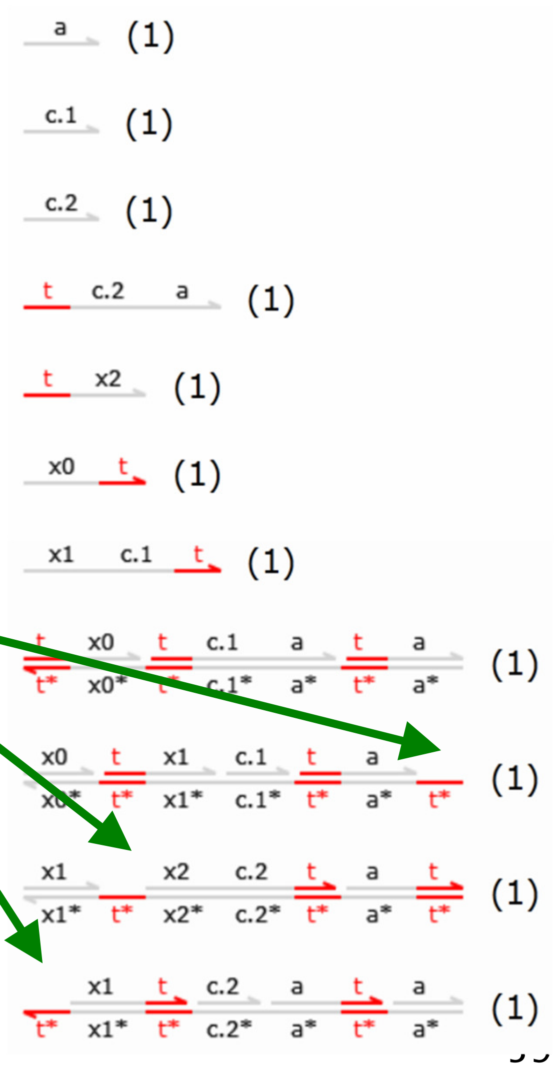


- Formalising correctness…
  - A [G "deadlock" => "all_done"]
  - E [F "all_done"]

Transducer flaw

- PRISM identifies a 5-step trace to the "bad" deadlock state
  - problem caused by "crosstalk" (interference) between DSD species from the two copies of the gates
  - previously found manually [Cardelli'10]
  - detection now fully automated
- Bug is easily fixed
  - (and verified)

reactive gates



Counterexample:
(1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
(0,1,1,0,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
(0,0,1,0,1,1,1,1,1,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
(0,0,1,0,1,1,1,1,0,0,1,1,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)
(0,0,1,0,1,1,0,1,0,0,1,1,1,0,0,0,1,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0)
(0,0,1,0,1,1,0,1,0,0,1,0,1,0,0,0,0,0,0,1,1,1,1,1,0,0,0,0,0,0,0,0)

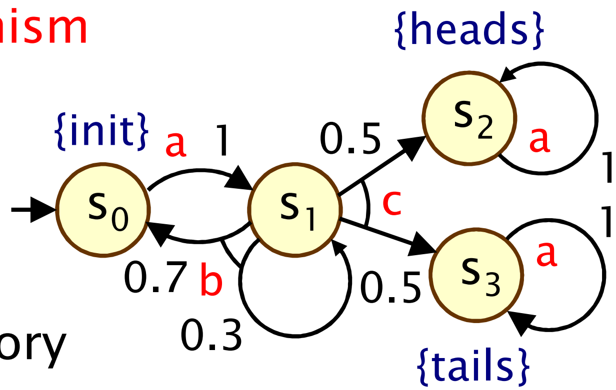
PRISM: Recent & new developments

- Major new features:
  1. multi-objective model checking
  2. parametric model checking
  3. real-time: probabilistic timed automata (PTAs)
  4. games: stochastic multi-player games (SMGs)

- Further new additions:
  - strategy (adversary) synthesis
  - CTL model checking & counterexample generation
  - enhanced statistical model checking
    (approximations + confidence intervals, acceptance sampling)
  - efficient CTMC model checking
    (fast adaptive uniformisation) [Mateescu et al., CMSB'13]
  - benchmark suite & testing functionality [QEST'12]
    www.prismmodelchecker.org/benchmarks/

1. Multi-objective model checking

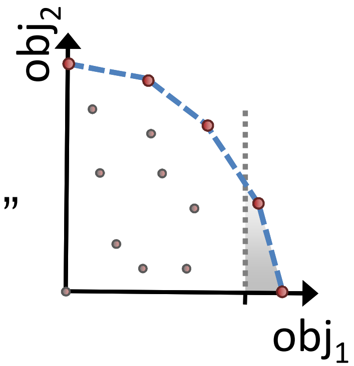
- Markov decision processes (MDPs)
  - generalise DTMCs by adding nondeterminism
  - for: control, concurrency, abstraction, …



- Strategies (or "adversaries", "policies")
  - resolve nondeterminism, i.e. choose an action in each state based on current history
  - a strategy induces an (infinite-state) DTMC
- Verification (probabilistic model checking) of MDPs
  - quantify over all possible strategies… (i.e. best/worst-case)
  - $P_{<0.01}$[F err] : “the probability of an error is always < 0.01”
- Strategy synthesis (dual problem)
  - "does there exist a strategy for which the probability of an error occurring is < 0.01?”
  - “how to minimise expected run-time?”

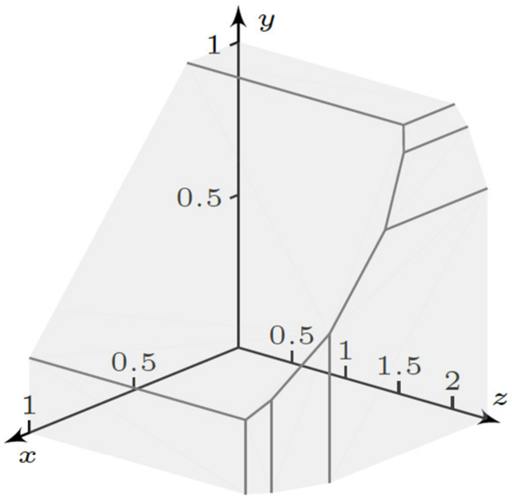
1. Multi-objective model checking

- Multi-objective probabilistic model checking
  - investigate trade-offs between conflicting objectives
  - in PRISM, objectives are probabilistic LTL or expected rewards
- Achievability queries
  - e.g. “is there a strategy such that the probability of message transmission is > 0.95 and expected battery life > 10 hrs?”
  - multi($P_{>0.95}$ [F transmit], $R^{time}_{>10}$ [C])
- Numerical queries
  - e.g. “maximum probability of message transmission, assuming expected battery life-time is > 10 hrs?”
  - multi($P_{max=?}$ [F transmit], $R^{time}_{>10}$ [C])
- Pareto queries
  - e.g. "Pareto curve for maximising probability of transmission and expected battery life-time”
  - multi($P_{max=?}$ [F transmit], $R^{time}_{max=?}$ [C])



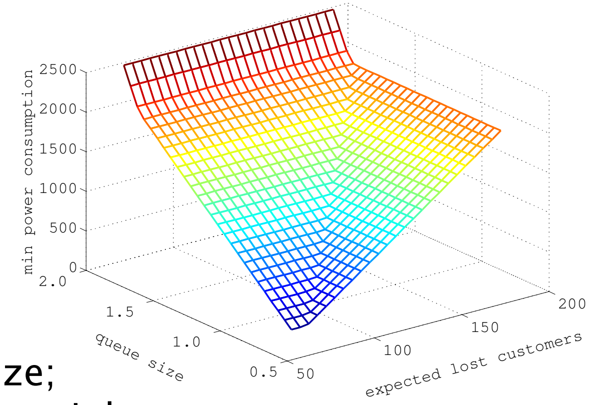
Multi-objective: Applications

Synthesis of team formation strategies [ATVA'12]



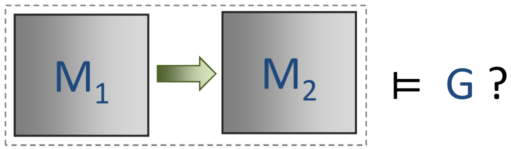
Pareto curve:
x="probability of completing task 1";
y="probability of completing task 2";
z="expected size of successful team"

Synthesis of dynamic power management controllers [TACAS'11]

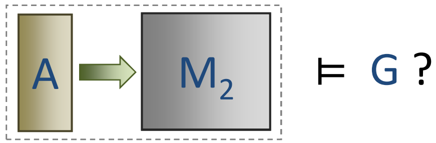


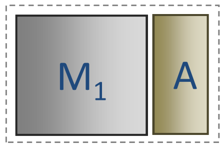
"minimise energy consumption, subject to constraints on:
(i) expected job queue size;
(ii) expected number of lost jobs

Probabilistic assume-guarantee framework [TACAS'10, TACAS'11]



Assume-guarantee query:
"does component M_2 satisfy guarantee G, provided that assumption A always holds?"
reduces to…
"is there an adversary (strategy) of M_2 satisfying A but not G?"



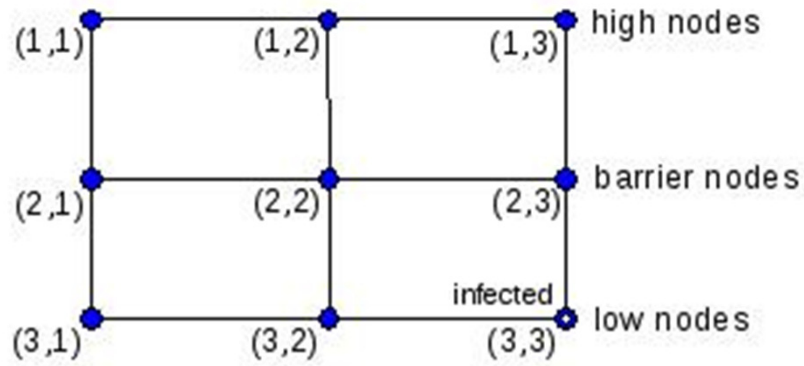


2. Parametric model checking

- Can specify models in parametric form [TASE13]
  - parameters expressed as unevaluated constants
  - e.g. const double x;
  - transition probabilities specified as expressions over parameters, e.g. 0.5 + x
- Properties are given in PCTL, with parameter constants
  - new construct constfilter (min, x1*x2, prop)
  - filters over parameter values, rather than states
- Determine parameter valuations to guarantee satisfaction of given properties
- Two methods implemented in PRISM (‘explicit’ engine)
  - constraints-based approach is a reimplementation of PARAM 2.0 [Hahn et al]
  - sampling-based approaches are new implementation

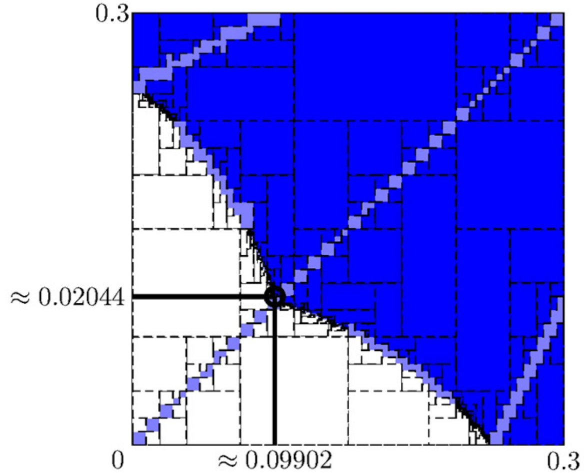
Case study: parametric network virus

- Parametric model of a network virus
  - a grid of connected nodes
  - virus spawns/multiplies
  - once infected, virus repeatedly tries to spread to neighbouring nodes
  - there are 'high' and 'low' nodes, with barrier nodes from 'high' to 'low'
  - choice of infection by virus probabilistic
  - choice of which node to infect nondeterministic



- Property specification
  - minimal expected number of attacks until infection of (1,1), starting from (N,N), is upper bounded by 20
  - probability of detection and of barrier nodes subject to repair by increasing p_{lhadd} and p_{baadd}

Case study: parametric models



Checking if minimal exp. number of attacks >= 20

Property constfilter(min,...,$R_{\{\text{“attacks”}\}>=20}$ [F “end”])

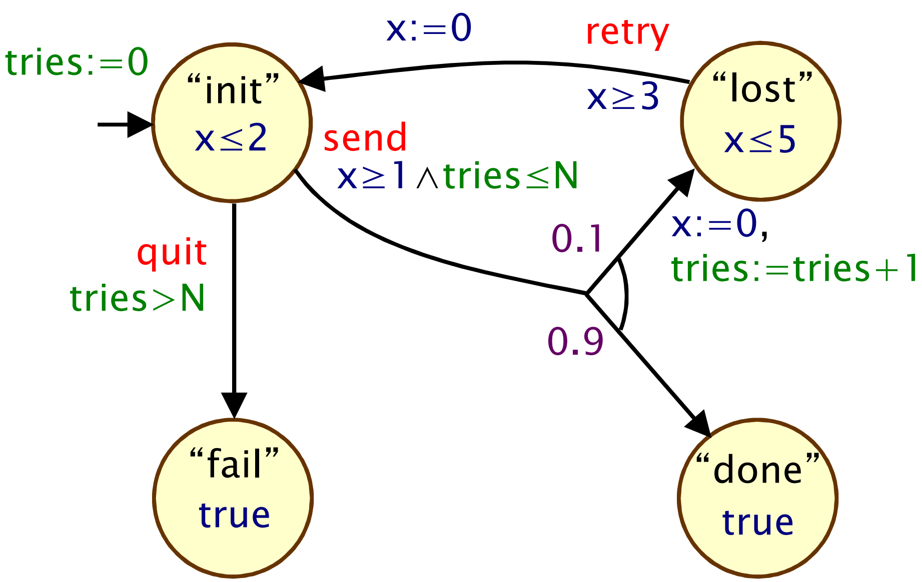
Model (network virus) has 809 states, ϵ = 0.05

Optimal value found in 2mins, showing optimal parameter values

3. Probabilistic timed automata (PTAs)

- Probability + nondeterminism + real-time
  - timed automata + discrete probabilistic choice, or…
  - probabilistic automata + real-valued clocks

- PTA example: message transmission over faulty channel



States
- locations + data variables

Transitions
- guards and action labels

Real-valued clocks
- state invariants, guards, resets

Probability
- discrete probabilistic choice

Modelling PTAs in PRISM

- PRISM modelling language
  - textual language, based on guarded commands

```
pta

const int N;

module transmitter

    s : [0..3] init 0;
    tries : [0..N+1] init 0;

    x : clock;

    invariant (s=0 ⇒ x≤2) & (s=1 ⇒ x≤5) endinvariant

    [send] s=0 & tries≤N & x≥1
          → 0.9 : (s'=3)
          + 0.1 : (s'=1) & (tries'=tries+1) & (x'=0);
    [retry] s=1 & x≥3 → (s' =0) & (x' =0);
    [quit]  s=0 & tries>N → (s' =2);

endmodule

rewards "energy" (s=0) : 2.5; endrewards
```

Modelling PTAs in PRISM

- PRISM modelling language
  - textual language, based on guarded commands

```
pta
const int N;
module transmitter
    s : [0..3] init 0;
    tries : [0..N+1] init 0;
    x : clock;
    invariant (s=0 ⇒ x≤2) & (s=1 ⇒ x≤5) endinvariant
    [send] s=0 & tries≤N & x≥1
           → 0.9 : (s'=3)
           + 0.1 : (s'=1) & (tries'=tries+1) & (x'=0);
    [retry] s=1 & x≥3 → (s' =0) & (x' =0);
    [quit]  s=0 & tries>N → (s' =2);
endmodule
rewards "energy" (s=0) : 2.5; endrewards
```

Basic ingredients:

- modules
- variables
- commands

Modelling PTAs in PRISM

- PRISM modelling language
  - textual language, based on guarded commands

```
pta
const int N;
module transmitter
    s : [0..3] init 0;
    tries : [0..N+1] init 0;
    x : clock;
    invariant (s=0 ⇒ x≤2) & (s=1 ⇒ x≤5) endinvariant
    [send] s=0 & tries≤N & x≥1
          → 0.9 : (s'=3)
          + 0.1 : (s'=1) & (tries'=tries+1) & (x'=0);
    [retry] s=1 & x≥3 → (s' =0) & (x' =0);
    [quit]  s=0 & tries>N → (s' =2);
endmodule
rewards "energy" (s=0) : 2.5; endrewards
```

Basic ingredients:

- modules
- variables
- commands

New for PTAs:

- clocks
- invariants
- guards/resets

Modelling PTAs in PRISM

- PRISM modelling language
  - textual language, based on guarded commands

```
pta
const int N;
module transmitter
    s : [0..3] init 0;
    tries : [0..N+1] init 0;
    x : clock;
    invariant (s=0 ⇒ x≤2) & (s=1 ⇒ x≤5) endinvariant
    [send] s=0 & tries≤N & x≥1
         → 0.9 : (s'=3)
         + 0.1 : (s'=1) & (tries'=tries+1) & (x'=0);
    [retry] s=1 & x≥3 → (s' =0) & (x' =0);
    [quit]  s=0 & tries>N → (s' =2);
endmodule
rewards "energy" (s=0) : 2.5; endrewards
```

Basic ingredients:

- modules
- variables
- commands

New for PTAs:

- clocks
- invariants
- guards/resets

Also:

- rewards (i.e. costs, prices)

Model checking PTAs in PRISM

- Properties for PTAs:
  - min/max probability of reaching X (within time T)
  - min/max expected cost/reward to reach X
    (for "linearly-priced" PTAs, i.e. reward gain linear with time)
- PRISM has two different PTA model checking techniques…
- "Digital clocks" – conversion to finite-state MDP
  - preserves min/max probability + expected cost/reward/price
  - (for PTAs with closed, diagonal-free constraints)
  - efficient, in combination with PRISM's symbolic engines
- Quantitative abstraction refinement
  - zone-based abstractions of PTAs using stochastic games
  - provide lower/upper bounds on quantitative properties
  - automatic iterative abstraction refinement

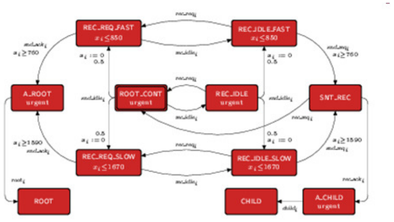
Case study: FireWire root contention

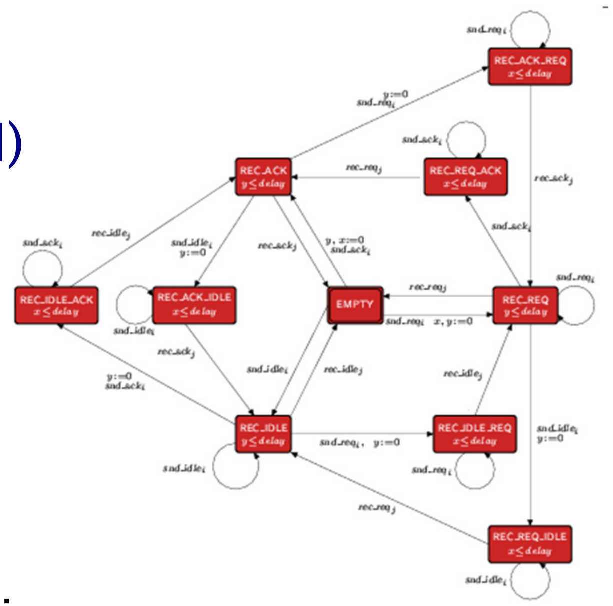
- FireWire (IEEE 1394)
  - high-performance serial bus for networking multimedia devices; originally by Apple
  - "hot-pluggable" - add/remove devices at any time
  - no requirement for a single PC (but need acyclic topology)

- Root contention protocol
  - leader election algorithm, when nodes join/leave
  - symmetric, distributed protocol
  - uses randomisation (electronic coin tossing) and timing delays
  - nodes send messages: "be my parent"
  - root contention: when nodes contend leadership
  - random choice: "fast"/"slow" delay before retry

Case study: FireWire root contention

- Detailed probabilistic model:
  - probabilistic timed automaton (PTA), including:
    - concurrency: messages between nodes and wires
    - timing delays taken from official standard
    - underspecification of delays (upper/lower bounds)
  - maximum model size: 170 million states



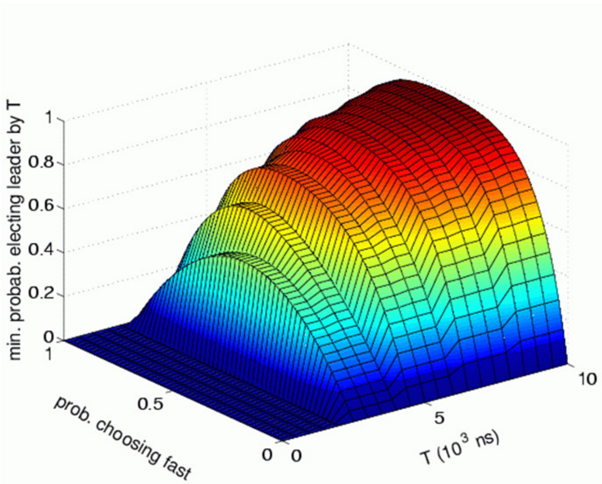


- Probabilistic model checking (with PRISM)
  - verified that root contention always resolved with probability 1
    - $P_{\geq 1}$ [F (end ∧ elected)]
  - investigated worst-case expected time taken for protocol to complete
    - $R_{max=?}$ [F (end ∧ elected)]
  - investigated the effect of using biased coin

Case study: FireWire root contention

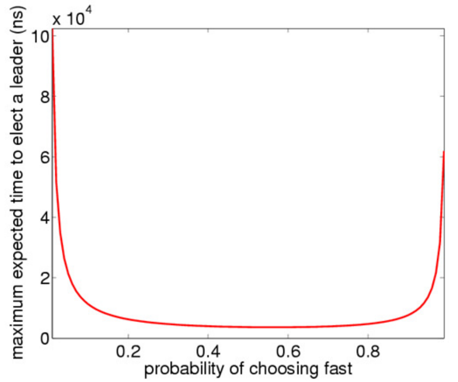
"minimum probability of electing leader by time T"

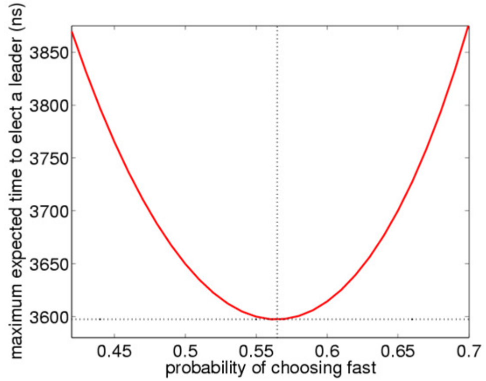
(using a biased coin)



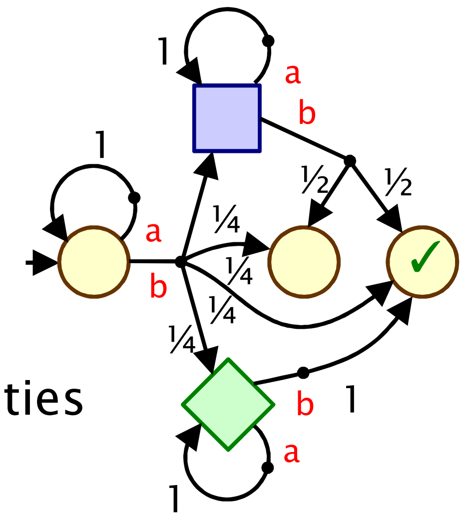
"maximum expected time to elect a leader"

(using a biased coin)





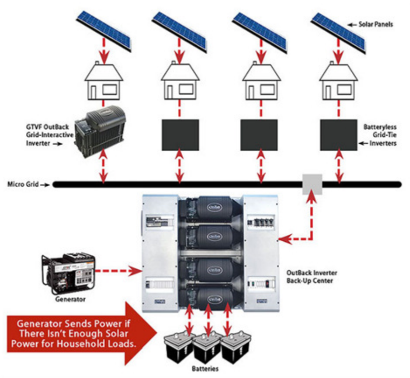
4. Stochastic multi-player games (SMGs)



- Stochastic multi-player games
  - players control states; choose actions
  - models competitive/collaborative behaviour
- Probabilistic model checking
  - automated methods to reason about complex player strategies and interaction with probabilities
- Property specifications
  - rPATL: extends Alternating Temporal Logic (and PCTL)
  - $\langle\langle\{1,3\}\rangle\rangle \, P_{<0.01} \, [\, F^{\leq 10} \, \text{error} \,]$
  - "do players 1 and 3 have a strategy to ensure that the probability of an error occurring within 10 steps is less than 0.01, regardless of the strategies of other players"
- Applications
  - controller synthesis (controller vs. environment), security (system vs. attacker), distributed algorithms, …
- PRISM-games: www.prismmodelchecker.org/games

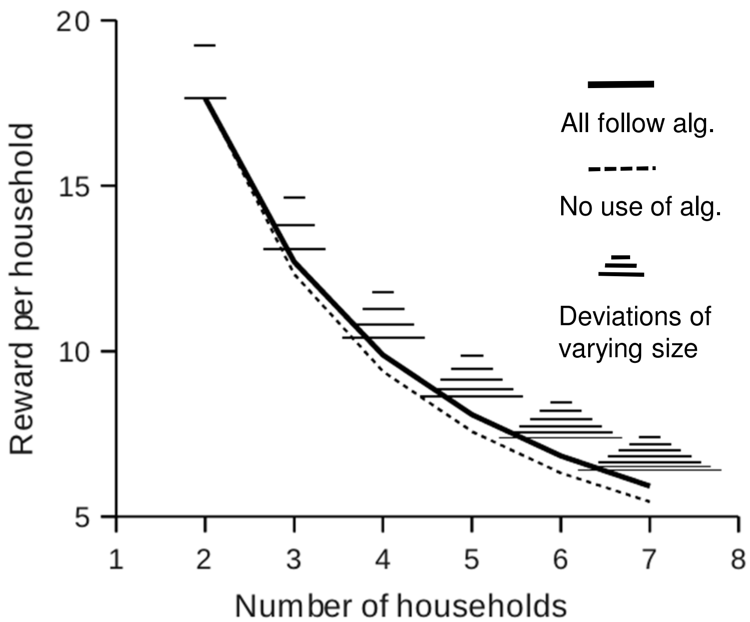
Case study: Energy management

- Energy management protocol for Microgrid
  - Microgrid: local energy management
  - randomised demand management protocol [Hildmann/Saffre'11]
  - probability: randomisation, demand model, …



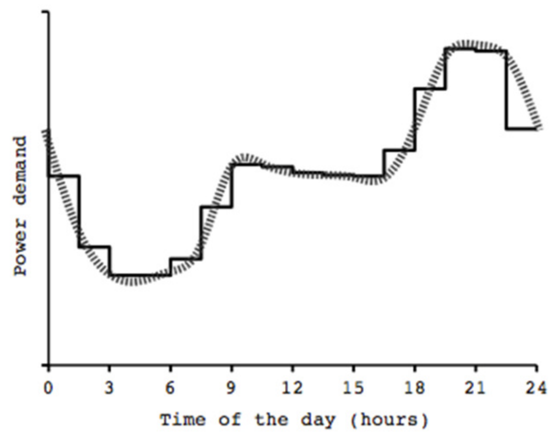
- Existing analysis
  - simulation-based
  - assumes all clients are unselfish

- Our analysis
  - stochastic multi-player game
  - clients can cheat (and cooperate)
  - exposes protocol weakness
  - propose/verify simple fix



Microgrid demand-side management

- The model
  - SMG with N players (one per household)
  - analyse 3-day period, using piecewise approximation of daily demand curve
  - add rewards for value V



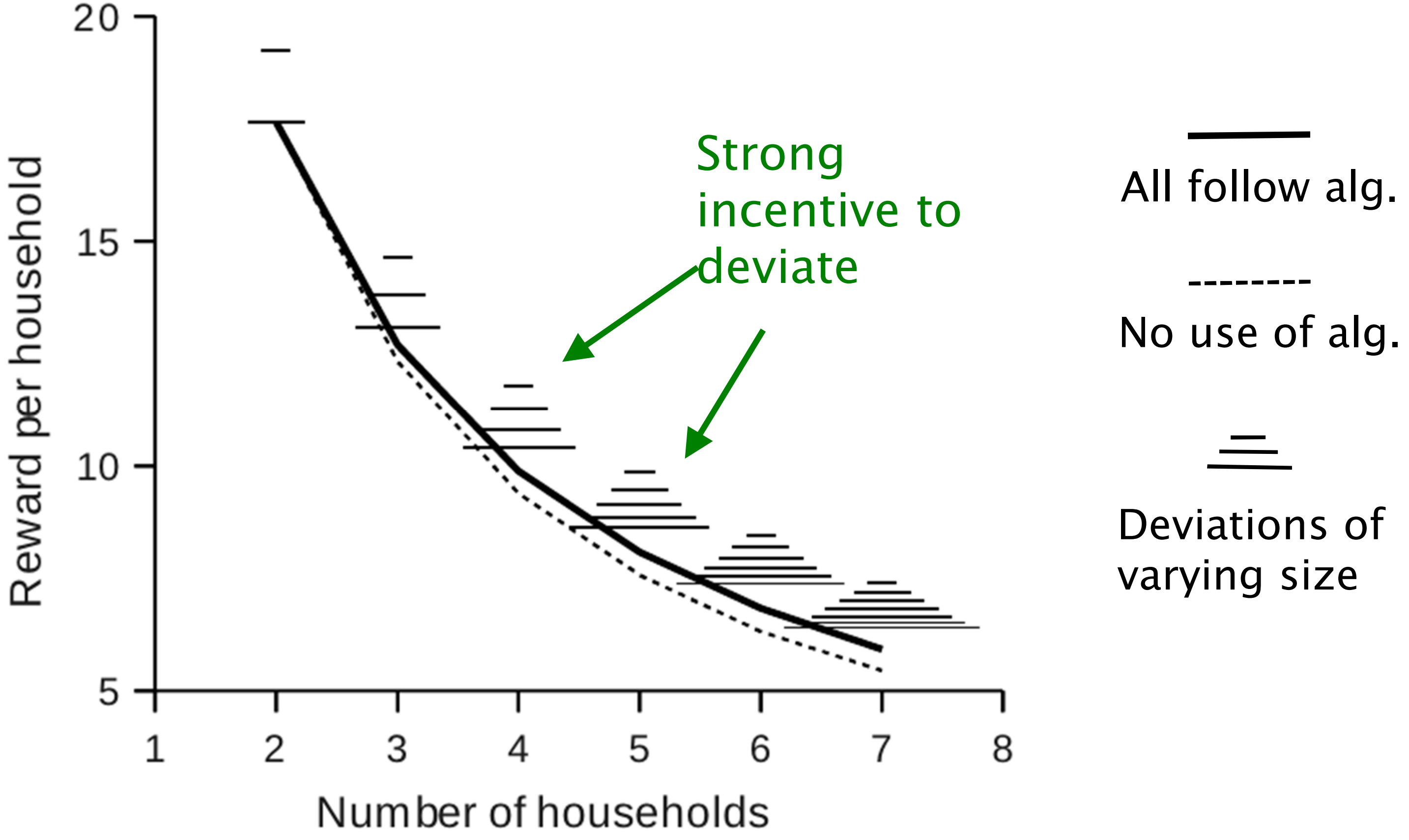
- Built/analysed models
  - for N=2,...,7 households

N	States	Transitions
5	743,904	2,145,120
6	2,384,369	7,260,756
7	6,241,312	19,678,246

- Step 1: assume all households follow algorithm of [HS'11] (MDP)
  - obtain optimal value for P_{start}
- Step 2: introduce competitive behaviour (SMG)
  - allow coalition C of households to deviate from algorithm

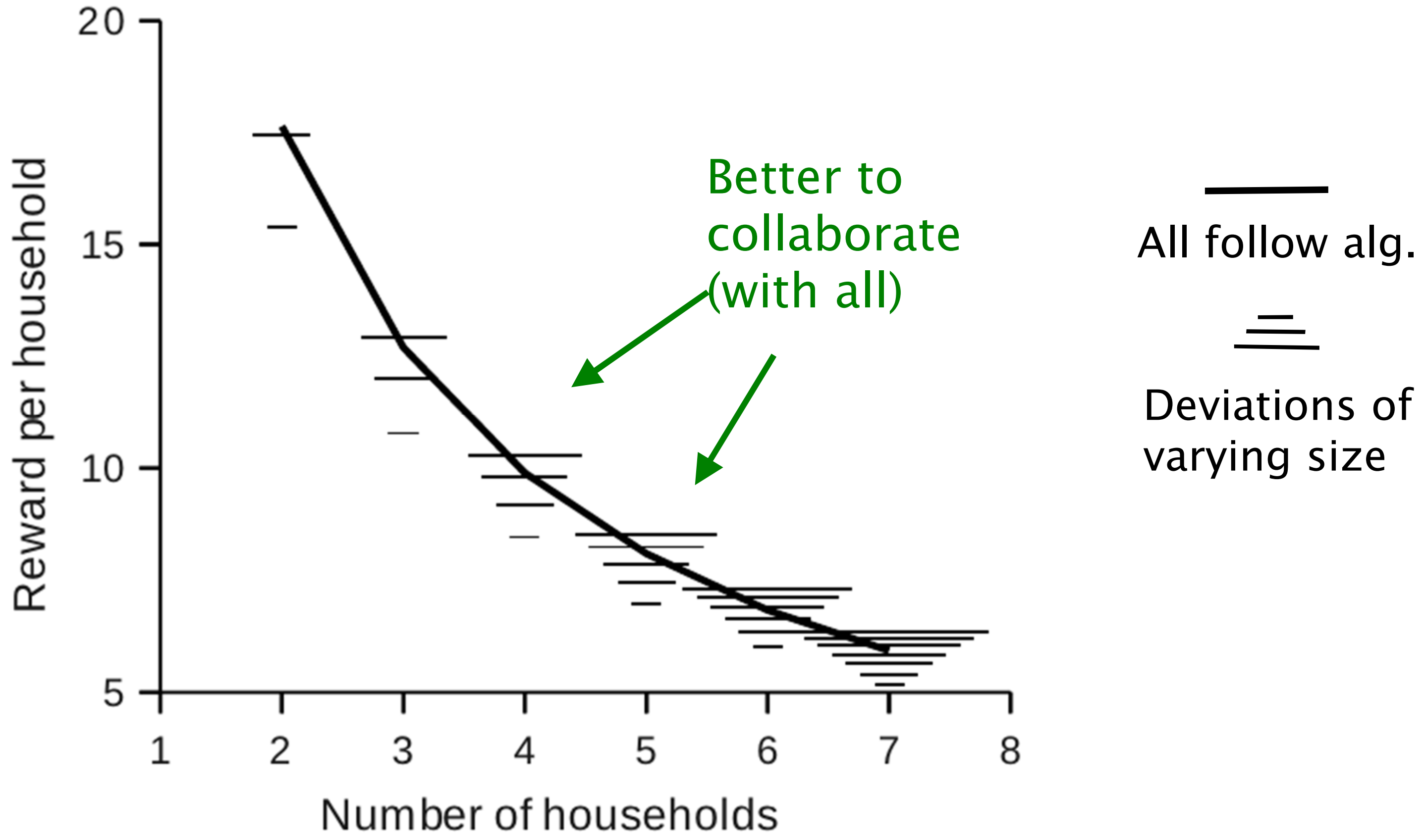
Results: Competitive behaviour

- The original algorithm does not discourage selfish behaviour…



Results: Competitive behaviour

- Algorithm fix: simple punishment mechanism
  - distribution manager can cancel some tasks



Conclusion

- Introduction to probabilistic model checking
- Overview of PRISM
- New developments
  1. multi-objective model checking
  2. parametric model checking
  3. real-time: probabilistic timed automata (PTAs)
  4. games: stochastic multi-player games (SMGs)
- Related/future work
  - quantitative runtime verification [CACM 2012]
  - statistical model checking [TACAS'04]
  - probabilistic hybrid automata [CPSWeek'13 tutorial]
  - autonomous urban driving [QEST'13]
  - verification of cardiac pacemakers [RTSS'12, HSCC'13]

References

- Tutorial papers
  - M. Kwiatkowska, G. Norman and D. Parker. *Stochastic Model Checking*. In SFM'07, vol 4486 of LNCS (Tutorial Volume), pages 220-270, Springer. June 2007.
  - V. Forejt, M. Kwiatkowska, G. Norman and D. Parker. *Automated Verification Techniques for Probabilistic Systems*. In SFM'11, volume 6659 of LNCS, pages 53-113, Springer. June 2011.
  - G. Norman, D. Parker and J. Sproston. *Model Checking for Probabilistic Timed Automata*. Formal Methods in System Design, 43(2), pages 164-190, Springer. September 2013.
  - M. Kwiatkowska, G. Norman and D. Parker. *Probabilistic Model Checking for Systems Biology*. In Symbolic Systems Biology, pages 31-59, Jones and Bartlett. May 2010.
- PRISM tool paper
  - M. Kwiatkowska, G. Norman and D. Parker. *PRISM 4.0: Verification of Probabilistic Real-time Systems*. In Proc. CAV'11, volume 6806 of LNCS, pages 585-591, Springer. July 2011.

Acknowledgements

- My group and collaborators in this work
- Project funding
  - ERC, EPSRC, Microsoft Research
  - Oxford Martin School, Institute for the Future of Computing

- See also
  - VERIWARE www.veriware.org
  - PRISM www.prismmodelchecker.org